

PROPERTIES OF HIGH-REDSHIFT LYMAN ALPHA CLOUDS

I. STATISTICAL ANALYSIS OF THE SSG QUASARS

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ABSTRACT

Techniques for the statistical analysis of the Lyman α forest in high redshift quasars are developed, and applied to the low resolution (25 Å) spectra of 29 of the 33 quasars in the Schneider-Schmidt-Gunn (SSG) sample. We extrapolate each quasar's continuum shortward of Lyman α emission, then consider each spectral bin of each quasar to be an (approximately) independent measurement of the absorption due to the Lyman α clouds. With several thousand such measurements thus available, we can obtain good determinations of some interesting properties of clouds in the redshift range $2.5 < z < 4.3$ without actually resolving any single cloud. We find that the mean absorption increases with z approximately as a power law $(1+z)^{\gamma+1}$ with $\gamma = 2.46 \pm 0.37$. The mean ratio of Lyman α to Lyman β absorption in the clouds is 0.476 ± 0.054 . We also detect, and obtain ratios, for Lyman β , γ , and possibly ϵ . We are also able to quantify the fluctuations of the absorption around its mean, and find that these are comparable to, or perhaps slightly larger than, that expected from an uncorrelated distribution of clouds. The techniques in this paper, which include the use of bootstrap resampling of the quasar sample to obtain estimated errors and error covariances, and a mathematical treatment of absorption from a (possibly non-uniform) stochastic distribution of lines, should be applicable to future, more extensive, data sets.

Subject headings: cosmology: observations – quasars – intergalactic medium

1. Introduction

Excepting only the quasars themselves, no observable objects have more potential for revealing the quantitative nature of the Universe at early times than do the Lyman α clouds, seen in absorption against the UV continuum of background quasars. Indeed, one might argue that the importance of the Lyman α clouds, at redshifts $z > 2.5$, in some respects exceeds that of the quasars, since the high-redshift quasars are “unusual” objects, presumably associated with the extreme statistical tail of structure formation, while the Lyman α clouds are probably more “typical” representatives of the state of baryonic matter, at least in the redshift range $2.5 < z < 4.3$.

From the time of their first discovery (Lynds 1971) and first detailed analysis (Sargent et al. 1980), the Lyman α clouds have principally been studied by the spectroscopic identification, and counting, of individual clouds along a quasar line of sight. (For a review, see, e.g., Sargent 1988a.) Since the cloud absorption lines are narrow, with equivalent widths on the order of 1 Å or less, observations with moderate-to-high spectral resolving power are required. Indeed, some exceptionally high quality QSO spectra (resolution $\lesssim 0.1\text{Å}$) have been obtained for use in direct studies of the cloud line profiles and velocity dispersion parameters (Pettini et al. 1990, Carswell et al. 1991).

At the opposite extreme lies the set of techniques that study the gross relative depression of the quasar continuum spectrum shortward of Lyman α (that is, emitted wavelength $\lambda_{em} < 1216\text{Å}$) by the aggregate effect of many clouds along the line of sight at redshifts smaller than the emission redshift z_{em} (Oke and Korycansky 1982; Bechtold et al. 1984; Schneider et al. 1989a,b, 1991; Giallongo and Cristiani 1990; Jenkins and Ostriker, 1991). Typical of these techniques is the association of a single number D_A with each quasar line of sight, the fractional absorption averaged over a broad band safely between the Lyman α and Lyman β emission wavelengths, say $1050\text{Å} < \lambda_{em} < 1170\text{Å}$.

Despite the obvious fundamental importance of understanding the nature of the intergalactic medium at high redshifts ($z > 3.5$, say), there have been very few high-redshift studies (Schneider et al. 1989b, 1991; Jenkins and Ostriker, 1991), and these have all used the D_A approach – because the quasars available for study are so faint.

In this paper we develop and apply a new technique, intermediate between the above extremes (though in most ways closer to the latter). Our data set consists of the carefully calibrated, low resolution spectra of 33 high-redshift quasars obtained by Schneider, Schmidt, and Gunn (1991; hereafter “SSG”). This sample provides the best available data set for $z > 3.5$ because (i) it contains all of the quasars with published redshifts $z > 3.85$ except PC 1247+3406 ($z = 4.897$, too large for this study), (ii) the spectra were

all obtained with the same instrumentation and thus have similar resolution and noise properties, and (iii) the sample contains some redshifts below $z = 3.5$, so there is some overlap with “detailed” studies. Our new approach (but cf. Webb et al. 1992) is to consider each separately resolved wavelength of each individual quasar to be an independent, albeit noisy, measurement of the cloud absorption in a particular three-dimensional volume of the Universe. (For $\lambda_{em} < 1025\text{\AA}$, the emission wavelength of Lyman β , each volume consists of more than one disjoint piece.)

We will find that, for the 25\AA spectral resolution of the SSG data, there are typically several or dozens, but not hundreds, of significantly absorbing clouds in each resolution element. Since, however, we have not 29 measured numbers, but rather several thousand, it is possible to untangle the statistics of the overlapping clouds to quite a remarkable degree. We will see, for example, that it is possible to obtain meaningful measurements of not only Lyman α absorption, but also Ly- β (1025\AA), Ly- γ (972\AA), Ly- δ (949\AA), and possibly Ly- ϵ (937\AA).

Jenkins and Ostriker (1991) showed that useful information could potentially be obtained from the detailed distribution of absorption values seen (at low or moderate resolution) in the Lyman α forest of a single quasar’s line of sight. We extend that important idea in this paper, and show that, from statistics of the forest’s fluctuating absorption, one can derive fairly precise measurements of several statistical quantities associated with the Lyman α cloud distribution.

The output of this paper is a set of techniques for measuring several quantitative statistical properties of the Lyman α cloud distribution, along with detailed uncertainty estimates (including cross-correlations), and also the results obtained by applying these techniques to the SSG sample quasars at redshifts $2.6 < z_{abs} < 4.2$. Paper II of this series will show that these measured values are already precise enough to impose strong constraints on the physical nature of individual clouds and on the distribution function of the cloud population. Using these derived observational constraints, Paper III will be able to confront the grander cosmological questions associated with the clouds: their origin, confinement mechanism, and implications for the formation of galaxies and large scale structure.

The focus of this paper, and later papers in this series, is on those Lyman α clouds at sufficiently high redshift to be considered (at least potentially) primordial cosmological objects, not associated with galaxies and minimally polluted by stellar element production. It is already clear (see, e.g, Sargent 1988b, Bahcall et al. 1992) that the detailed properties of quasar absorption clouds at low redshifts may differ significantly from those at high redshifts. We do not expect the results of this paper to apply at low redshifts; the question

“how low is low?”, obviously an interesting one, will be considered in a later paper.

In this paper, §2 discusses how we fit the individual quasar spectra to obtain the underlying continua. In §3 we estimate the mean optical depth in Lyman α absorption as a function of redshift z . §4 extends this estimate to the other accessible Lyman lines. §5 analyzes the fluctuation statistics of the absorption measurements. §6 summarizes our conclusions.

In Appendix A we analyze in detail a class of statistical models for the distribution of spectral lines in the Lyman forest. These are similar to the usual Poisson processes, but allow for a nonuniform underlying density of points. We obtain results for the mean of the transmission coefficient and also for its correlation function. In Appendix B the results of Appendix A are specialized for the particular applications of this paper.

2. Estimating the Continuum

The input data to this study are the calibrated SSG spectra of 33 quasars in the redshift range of $3.1 < z_{em} < 4.8$. The data are in the form of corrected (for atmospheric absorption, reddening, and instrumental response) fluxes, in 10 Å bins, from 4310 Å to 9500 Å observed wavelength. The resolution of the measurements is about 25 Å, so neighboring bins are not independent. (We make use of this oversampling below.) Details of the observations and data reduction procedures are described in SSG.

Three of the SSG quasars are broad absorption line (BAL) quasars. We eliminate these from the sample. At a later stage of our processing, we also eliminate one additional quasar, the highest redshift member of the sample, because its available data do not adequately determine certain fitted parameters (see below). The 29 remaining quasars, along with their redshifts and monochromatic AB magnitudes at emitted wavelength 1450 Å as determined by SSG, are listed in Table 1. (Emitted wavelength 1450 Å, in the gap between the SiIV/OIV blend at 1400 Å and CIV at 1549 Å, is chosen as a relatively clean measurement of the underlying quasar continuum.)

While quasar studies frequently focus on differences among quasar spectra, for our purposes it is equally important to take note of the similarity of all the spectra in this sample. Figure 1 plots all of the SSG measurements of all 29 quasars, as a function of emitted wavelength in the range 930 Å to 2200 Å, normalized to an (arbitrary) common AB_{1450} magnitude of 18. One sees clearly all of the spectral features present in low redshift

<i>Quasar</i>	z	AB_{1450}	<i>Quasar</i>	z	AB_{1450}
PC2344+0124	3.143	19.10	PC2047+0123	3.799	19.23
PC0056+0125	3.149	18.41	PC1643+4631B	3.831	20.35
PC1601+3754	3.188	19.68	PC1301+4747	4.004	21.32
PC2132+0126	3.194	19.69	Q0046–293	4.014	19.26
PC1605+4631	3.203	19.79	PC0910+5625	4.035	20.86
PC0118+0119	3.241	19.19	Q0101–304	4.072	19.98
PC2226+0216	3.273	18.80	PC2331+0216	4.093	19.84
PC0234+0120	3.300	19.96	Q0000–26	4.098	17.46
PC0344+0222	3.377	20.09	PC0104+0215	4.171	19.67
PC1619+4631	3.471	20.54	PC0751+5623	4.281	19.65
PC1548+4637	3.544	19.07	PC0307+0222	4.379	19.92
PC0345+0130	3.638	19.49	Q2203+29	4.399	20.41
PC1640+4628	3.700	19.29	PC1233+4752	4.447	20.11
PC1643+4631A	3.790	20.05	PC0953+4749	4.457	19.09
PC0131+0120	3.792	19.08			

Table 1: The 29 quasars studied in this paper. Redshifts and corrected continuum AB magnitudes at emitted wavelength 1450 Å are from Schneider, Schmidt, and Gunn (1991). Four SSG quasars, the three broad absorption line quasars, and quasar PC1158+4635 at redshift $z = 4.733$, are omitted from this study (see text).

quasars (see, e.g., Francis, et al. 1991). The Lyman α depression shortward of 1216 Å is seen with equal clarity.

While the logarithmic slope of the underlying continuum indisputably varies from quasar to quasar, it is striking, in Figure 1, that this is a relatively small effect over the wavelength range shown. The reliability of our results will depend somewhat on the accuracy with which we are able to extrapolate each quasar’s underlying continuum down to ~ 930 Å. Figure 1 shows that this extrapolation, amounting to less than 25% of the range of abscissa shown, is not so daunting as one might think.

To fit for the continua, we use all the available measurements between $\lambda_{em} = 1250$ Å and $\lambda_{em} = 2200$ Å. For each quasar in the sample, we proceed as follows. First, we estimate a relative statistical error associated with each measured data point. This is done by (a) calculating the variance of each point with its immediately adjacent neighbors, and (b) convolving this series of raw variances with a Gaussian profile of width about 12 bins (FWHM). (Here is where we exploit the fact that the data is oversampled.) It is not important that this estimate of statistical error be correctly normalized, but only that it reflect, in a general way, the relative weight to be assigned to individual measurements in the next stage of fitting.

Now, we fit the data to the following 22-parameter *linear* model in the emitted rest frame,

$$F(\lambda) = C_{1/2}\lambda^{1/2} + C_1\lambda + \sum_{i=1}^{10} \left[A_i + B_i \left(\frac{\lambda - \lambda_i}{2w_i} \right)^2 \right] \exp \left[- \left(\frac{\lambda - \lambda_i}{2w_i} \right)^2 \right] \quad (1)$$

for the parameters $C_{1/2}$, C_1 , $A_1 \dots A_{10}$ and $B_1 \dots B_{10}$. Parameters $C_{1/2}$ and C_1 characterize the underlying continuum with two degrees of freedom; over our limited wavelength range they contain information equivalent to a magnitude and a spectral index. The constant values λ_i and w_i , $i = 1, \dots, 10$, are the wavelengths and *nominal* widths of the 10 fitted lines, and are given in Table 2. Parameters $A_1 \dots A_{10}$ are the fitted strengths of the emission lines, while $B_1 \dots B_{10}$ parametrize the ratio of the fitted width to the nominal width. The nominal widths in Table 2 were obtained by fitting a nonlinear model to the composite spectrum of Figure 1.

The functional form of equation (1) is perhaps slightly unconventional, and is motivated by the desire that it be linear in all parameters. More conventionally, one might fit nonlinearly for a continuum magnitude and spectral index and for the intensities and widths of Gaussian-profile lines. However, with the present noisy data, and the virtual certainty that there are other features in the data besides those modeled (especially longward of 1700 Å), we found such nonlinear fits to be quite fussy and to require

considerable user intervention. Since we are not here interested in details of the shapes of lines in their wings, but we *are* interested in treating the entire data set in as statistically homogeneous manner as possible, the relative robustness of a purely linear fit is desirable.

It is at this stage that we eliminate quasar PC1158+4635 at redshift $z = 4.733$ from the sample: Because of the large redshift, its longer wavelength emission features are not well measured, and the above fit is ill determined. While we could easily fit its continuum with a smaller number of parameters, we instead choose to preserve the homogeneity of processing procedure by eliminating the quasar entirely.

Figure 2 shows all the data in the rest frame range 1250 Å to 1800 Å, and the fitted model spectra, individually for the 29 quasars. Also plotted for each quasar in the figure is the adopted underlying continuum. In many cases this is simply the function

$$F(\lambda) = C_{1/2}\lambda^{1/2} + C_1\lambda \quad (2)$$

where the C 's are fitted parameters. In other cases, however, while the full model of equation (1) is quite well determined, its dissection into equation (2) plus a remainder is quite degenerate numerically, as evidenced by a nearly degenerate family of fits that trade off variations in $C_{1/2}$ and C_1 against unphysical (e.g., negative) values for the line strengths. In these cases we refit (essentially by eye, although the process could be automated) a continuum of the form of equation (2) to the output fitted curve of (1), requiring the continuum to lie below the model at selected wavelengths between emission lines. In virtually all cases this refitting is quite well determined, suggesting that one could readily replace our procedure by a single linear fit with positivity constraints; we have not, however, implemented this.

The relation between equation (2) and the index α of the more familiar power-law continuum model,

$$f_\nu \propto \nu^\alpha \quad (3)$$

at some fiducial wavelength λ_0 is

$$\alpha = -\frac{0.5C_{1/2} + C_1\lambda_0^{1/2}}{C_{1/2} + C_1\lambda_0^{1/2}} \quad (4)$$

We find a good correlation (e.g., at $\lambda_0 = 1450\text{\AA}$) between the spectral indices thus obtained and those reported in SSG; however, our α values are systematically larger (smaller in magnitude) by a few tenths, with a mean in the range -0.5 to -0.6 reported by Richstone and Schmidt (1980), Steidel and Sargent (1987), Warren et al. (1991), and others.

Adopting these continuum models, and extrapolating them from 1250 Å down to 930 Å, we obtain the results shown, individually by quasar, in Figure 3. That figure shows all

SSG measurements shortward of emitted wavelength 1200 Å in relation to their respective extrapolated continua. All of the rest of this paper is derived from these data. No further use is made of the data longward of 1200 Å.

3. Mean Optical Depth as a Function of Redshift

The emitted wavelength range 1050 Å to 1170 Å is substantially uncontaminated by either Lyman α or Lyman β emission, and also (since it is longward of Lyman β) uncontaminated by Lyman β absorption by intervening clouds. While this range could in principle be contaminated by C IV and Mg II absorption (see, e.g., Meyer and York 1987), or by quasar emission lines in this wavelength region (but see below for evidence against this), we will adopt the conventional assumption that these effects are likely negligible at high redshifts; probably one is seeing a virtually pure sample of Lyman α absorption by clouds. The data in this range from the 29 SSG quasars are plotted in Figure 4. The ordinate is the ratio of observed flux to extrapolated continuum, i.e., the fractional transmitted flux, or transmission. The abscissa of the left panel is emitted wavelength. One sees a widely scattered distribution of points. The abscissa of the right panel is observed wavelength or (equivalently, see top scale) absorption redshift. Here one sees a strong trend with redshift. Note that the right panel plots only data points which satisfy the wavelength cuts of the left panel, not the vastly larger number of individual SSG measurements with observed wavelength in the range shown (4300 Å to 6500 Å).

There are in fact a few data points (not shown) with transmission greater than 1, i.e., with observed fluxes above the extrapolated continuum, and a few points less than zero, i.e., where SSG’s background subtraction gives negative results. These are obviously defective values; to eliminate some less obvious, but likely defective, values, we adopt slightly tighter cuts on the data, and accept points with transmission between 0.1 and 0.9. (We have verified that our results are not sensitive to the values of these cutoffs.) There are 1596 surviving points in the sample.

Converting transmission to an equivalent optical depth (by taking the negative logarithm), we fit the data shown in Figure 4 a model of the conventional form

$$\bar{\tau}_\alpha(z) = A(1+z)^{\gamma+1} \quad (5)$$

and obtain the best-fit values $A = 0.0037$ and $\gamma = 2.46$. The reason for defining the exponent to be $\gamma + 1$ is so that our γ is directly comparable to its conventional usage as the exponent in the number distribution of clouds (see equations 2.1–2.2 of Jenkins and Ostriker 1991).

3.1. The Bootstrap Resampling Method

As a digression, we here need to discuss in some detail how we obtain error estimates (or variances) on the quantities A and γ , since we will follow a similar paradigm in obtaining variances and covariances for various further quantities in later sections. There are various reasons why the formal errors that come out of the fitting procedure are meaningless: We don't have good error estimates on the individual measurements. (The error estimates used in fitting the continuum were relative, not absolute.) The individual measurements shown in Figure 4 are not statistically independent, both because there is more than one measurement per resolution bin (adjacent measurements sample overlapping cloud populations), *and* because the points associated with a single quasar share a common source of error in the determination of that quasar's extrapolated continuum.

A seemingly simple, yet very powerful, method for estimating the errors is by the statistical bootstrap (or resampling) method (Efron 1982, Efron and Tibshirani 1986, Press et al. 1992). From the sample of 29 quasars, we generated repeated *resampled* sets of 29 quasars by drawing randomly *with replacement*. The typical set will thus have on the order of 12 duplicated quasars. No matter; we repeat exactly the data reduction which produced the original determinations of A and γ . The population of resulting values for A and γ can be shown (with certain technical assumptions which need not concern us) to be distributed around the original determinations in the same way, both variance and covariance, that the original determinations ought to be distributed with respect to the true value.

More generally, and to be applied below, suppose that we have some statistical procedure that determines the M quantities R_i , $i = 1, \dots, M$, and suppose that we make N resamplings. We denote the k th determination of the i th quantity by R_i^k , $k = 0, \dots, N$, where $k = 0$ uses the full sample and $k = 1, \dots, N$ are the resamplings. Then the best estimates for the R_i 's are given by R_i^0 , $i = 1, \dots, M$. The standard errors $\sigma(R_i)$ are estimated by

$$\sigma^2(R_i) \approx \frac{1}{N} \sum_{k=1}^N (R_i^k - \bar{R}_i)^2 \quad (6)$$

where

$$\bar{R}_i \equiv \frac{1}{N} \sum_{k=1}^N R_i^k \quad (7)$$

The correlation matrix C_{ij} among the R_i^0 's is estimated by

$$C_{ij} \approx \frac{1}{N} \sum_{k=1}^N (R_i^k - \bar{R}_i)(R_j^k - \bar{R}_j) \quad (8)$$

An important quantity is the inverse correlation matrix C_{ij}^{-1} , since a set of model predictions R_i^* for the quantities can be compared to the measured values R_i^0 by calculating

$$\chi^2 = \sum_{i,j=1}^M (R_i^* - R_i^0) C_{ij}^{-1} (R_j^* - R_j^0) \quad (9)$$

which should be chi-square distributed with M degrees of freedom (see, e.g., Rybicki and Press 1992). Another sometimes useful set of quantities are the coefficients of correlation among the individual measurements, given by $r_{ij} \equiv C_{ij}/(\sigma_i\sigma_j)$.

We can now return to our particular example: To obtain error bars on our measurement of γ and A , we performed 100 resamplings, obtaining the results

$$\gamma = 2.46 \pm 0.37 \quad A = 0.0037 \pm 0.0024 \quad (10)$$

where the errors are 1σ . The parameters A and γ turn out to be very highly correlated. If we consider γ as the more fundamental quantity, then almost all of the error in A can be moved to the determination of γ , as

$$A = 0.0175 - 0.0056\gamma \pm 0.0002 \quad (11)$$

The solid line in Figure 4 shows the mean absorption of equations (5)–(10), transformed back to an ordinate of transmission. The shaded band shows the result of changing γ by $\pm 1\sigma$, with A simultaneously changed by equation (11).

The values of A and γ that we obtain are consistent with the results of Jenkins and Ostriker (1991), who measure an aggregate continuum depression D_A for each quasar sampled, and in good agreement with previous determinations which relied on the counting of individual lines. For example, Murdoch et al. (1986) obtained $\gamma = 2.31 \pm 0.40$; Bajtlik et al. (1988) obtained 2.36 ± 0.40 ; Lu, Wolfe, and Turnshek (1991) obtained 2.37 ± 0.26 . What is new in this investigation is the extension to higher redshifts (Table 1), and the use of a large quantity of low resolution data without line counting and without aggregation into broad spectral bands. Note that the large values of γ at high redshift are quite different from those observed at low redshift (e.g., Bahcall et al. 1993). In §5, below, and in Paper II, we will see the additional information that comes from an absolute determination of the constant A , and from our ability (not present in aggregate determinations of D_A) to look at the distribution of individual points around the fitted mean values.

4. Relative Absorption Strengths of Lyman β , γ , δ , and ϵ

In Figure 5 we replot all the data shortward of $\lambda_{em} = 1200\text{\AA}$, but now removing (individually for each datum at its own appropriate absorption redshift) the effect of the mean Lyman α absorption, equations (5) and (10). The ordinate is the same as Figure 4, namely ratio of observed flux to the extrapolated continuum. The shaded band in the figure is simply a moving-window average; its fluctuation scale is simply an artifact of the chosen window size.

One sees that, shortward of the remaining tail of Lyman α emission and down to 1025\AA , the corrected fluxes indeed scatter around flux ratio unity. There is clear O VI emission around 1032\AA , with perhaps some contribution from Lyman β , then a visible Lyman β decrement extending down to Lyman γ at 972\AA . The γ and δ decrements are also visible, although we will need more definitive statistics (below) to quantify their significance. The decrement below Lyman ϵ at 937\AA while present in the data, will turn out to be of questionable statistical significance in this sample.

We can now quantify the observed relative absorptions of Lyman β , γ , δ , and perhaps ϵ , relative to Lyman α ; in other words the ratios of the mean equivalent widths (the precise definition of which is equation 16 below) of different Lyman absorption lines in the clouds: W_β/W_α , W_γ/W_α , W_δ/W_α , etc. In the limited redshift range that we are studying, we will assume that the equivalent width ratios do not depend on redshift. These ratios are, as we will see in Paper II, direct indicators of the physical state of the clouds.

An important point is that we are *not* simply quantifying the decrements seen in Figure 5. That figure does not display the fact that each datum has an individual emission redshift z_{em} associated with it. Use of these individual redshifts (through equation 5) allows, at least in principal, the removal of that part of the scatter in Figure 5 that is due to the range of redshifts in the quasar sample. For example, W_β/W_α is estimated using all points with λ_{em} between 972\AA and 1020\AA . Each observed point generates a one-point estimate of the form

$$\frac{W_\beta/\lambda_\beta}{W_\alpha/\lambda_\alpha} \approx \frac{\ln(Q^{-1}) - \bar{\tau}_\alpha(\lambda_{obs}/1216\text{\AA} - 1)}{\bar{\tau}_\alpha(\lambda_{obs}/1025\text{\AA} - 1)} \quad (12)$$

where Q is the ratio of observed flux to the extrapolated continuum of the point (i.e., the observed transmission), and $\bar{\tau} = \bar{\tau}(z_{abs})$ is given by equation (5). Here, $\lambda_\alpha = 1216\text{\AA}$ and $\lambda_\beta = 1025\text{\AA}$ are the laboratory wavelengths. The one-point estimates are then averaged.

To estimate W_γ/W_α , we use points in the range λ_{em} between 949\AA and 972\AA . Now we must subtract from the measured total optical depth $\ln(Q^{-1})$ both Lyman α and Lyman β corrections at their respective absorption redshifts and relative strengths,

$$\frac{W_\gamma/\lambda_\gamma}{W_\alpha/\lambda_\alpha} \approx \frac{\ln(Q^{-1}) - \bar{\tau}_\alpha(\lambda_{obs}/1216\text{\AA} - 1) - \bar{\tau}_\beta(\lambda_{obs}/1025\text{\AA} - 1)}{\bar{\tau}_\alpha(\lambda_{obs}/972\text{\AA} - 1)} \quad (13)$$

where

$$\bar{\tau}_\beta(z_{abs}) \equiv \left(\frac{W_\beta/\lambda_\beta}{W_\alpha/\lambda_\alpha} \right) \bar{\tau}_\alpha(z_{abs}) \quad (14)$$

is the derived mean optical depth in Lyman β as a function of redshift.

One continues in this manner to obtain the ratio W_δ/W_α using points in the range 937 Å to 949 Å, and W_ϵ/W_α using points in the range 930 Å to 937 Å. Obviously, looking at Figure 5, one must at some point begin to wonder whether the values obtained have any meaning. Also, it is clear that, because of the repeated subtraction of earlier ratios, errors in the determination of later ratios will be highly correlated (actually, anticorrelated) with the errors of earlier ratios.

Once again, the bootstrap technique of resampling with replacement provides quantitative answers to these concerns. As for the previously determined parameters A and γ , we have made 100 resampled determinations of the 4 ratios W_β/W_α , W_γ/W_α , W_δ/W_α , and W_ϵ/W_α , choosing a different set of 29 quasars in each resampling.

In the notation of the discussion leading to equation (6), let R_i , $i = 1, \dots, 4$, now denote the above four ratios, in the obvious order. Then the resulting measured quantities R_i^0 , $\sigma(R_i)$, C_{ij} , C_{ij}^{-1} , and r_{ij} are given in Table 3. Although the last ratio, W_ϵ/W_α , is not well determined (detected at less than two standard deviations), the other ratios seem quite well established by the data. The first two ratios, W_β/W_α and W_γ/W_α , are determined by this data set with about 20% accuracy. We will see in Paper II that these values are able to constrain the nature of Lyman α clouds at high redshifts that are not yet accessible to high resolution studies.

5. Fluctuations in Optical Depth around Mean Values

In principle there is information not only in the mean values of Lyman α absorption (and the higher lines β , γ , δ), but also in the statistics of the fluctuations of individual points around the mean. In Figure 5, for example, one wants to identify the mechanisms that contribute to the spread of the individual points around the mean, defined in that Figure to be unity between 1050 Å and 1170 Å. Measurement noise, which tells us nothing about the Lyman α clouds, is one contributing factor, as are uncertainties in our extrapolation of the quasar continua, which are also not of intrinsic interest.

The more interesting contribution, because it potentially does give information about the clouds, is that of the Poisson statistics of how many clouds are present in each spectral resolution element.

Unfortunately, as we will now see, the exploitation of this information is not straightforward.

Let us first revisit the question of the mean transmission. The absorption equivalent width W of a cloud (in its rest frame) depends on its physical parameters, for example column density N , temperature parameter b , and so on. Let \mathbf{p} denote the vector of such parameters, and let $\mathcal{N}(\mathbf{p})d\mathbf{p}$ be the number of clouds along the line of sight *per unit rest wavelength* in a volume $d\mathbf{p}$ of parameter space. Then the total number of clouds per unit rest wavelength is given by

$$n = \int \mathcal{N}(\mathbf{p})d\mathbf{p} \quad (15)$$

(if the integral converges), while the mean equivalent width of a cloud is

$$\overline{W} = \frac{1}{n} \int W(\mathbf{p})\mathcal{N}(\mathbf{p})d\mathbf{p} \quad (16)$$

If the lines are randomly placed, e.g., without clustering, then the mean transmission \overline{Q} is given by

$$\overline{Q} = \exp(-n\overline{W}) \quad (17)$$

if the integral for n exists, and

$$\overline{Q} = \exp\left(-\int W(\mathbf{p})\mathcal{N}(\mathbf{p})d\mathbf{p}\right) \quad (18)$$

otherwise (for example, if there are an infinite number of lines with negligibly small W). Equation (17) is so natural as to seem intuitively obvious, but it is in fact quite a nontrivial result, relating the average of a (nonlinear) exponential to the average of the exponent. A proof is given by Goody (1964, §4.5); see also the Appendices, equations (A15), (B24), and (B25).

The sad fact is that there is no such universal result for the next moment,

$$\text{Var}(Q)/\overline{Q}^2 = \left\langle \left(\frac{Q}{\overline{Q}} - 1 \right)^2 \right\rangle \quad (19)$$

Rather, the variance $\text{Var}(Q)/\overline{Q}^2$ depends in a nontrivial way on the joint distribution of equivalent widths (or, more fundamentally, column density N) and doppler widths b , as well as on the instrumental spectral resolution.

Some simple model cases are derived in Appendix B and are illuminating. Assume a square line profile of full width W_0 , and suppose that all lines have the same column density,

and thus the same equivalent width \overline{W} . Let Δ be the instrumental spectral resolution. Then an exact result and its power series expansion are

$$\text{Var}(Q)/\overline{Q}^2 = \frac{2W_0}{\Delta} \left[\frac{1}{A}(e^A - 1) - 1 \right] \approx \frac{n\overline{W}^2}{\Delta} \left(1 + \frac{A}{3} + \dots \right) \quad (20)$$

where

$$A \equiv \frac{n\overline{W}^2}{W_0} \quad (21)$$

See Appendix B, equations (B41) and (B44). If the instrumental response is not square but is rather described by a response function $w(\lambda - \lambda_0)$, then the definition of Δ is

$$\Delta = \left(\int w d\lambda \right)^2 / \int w^2 d\lambda \quad (22)$$

(cf. equation B30 in Appendix B).

The limiting form of equation (20) for small A can be reproduced, up to a constant, by a simple physical argument: Because the lines have a width W_0 , the instrumental width Δ contains Δ/W_0 independent elements. In each of these, the mean optical depth is $n\overline{W}$, while the number of lines contributing to this mean is nW_0 . Thus, in each element, the r.m.s. fluctuation in optical depth, which is also the fractional fluctuation in transmission Q is about $(n\overline{W})(nW_0)^{-1/2}$. Averaging over the independent elements reduces this fractional fluctuation by an additional factor $(W_0/\Delta)^{1/2}$, yielding finally

$$\text{Var}(Q)/\overline{Q}^2 \sim \frac{n\overline{W}^2}{\Delta} \quad (23)$$

The reason that the exact result (20) has an additional exponentially increasing factor in A is that, for large optical depths and saturated lines, the fluctuations become dominated by the cases where increasingly rare “windows” happen to occur in the random placement of the lines.

A second analytic result derived in Appendix B (cf. equations B41 and B48) is for the case where there is no characteristic value for \overline{W} , but rather a power-law distribution in N , the column density. (This is a more realistic idealization of the actual situation for the Lyman α clouds.) If the number of clouds along the line of sight with N between N and $N + dN$ is proportional to $N^{-\beta} dN$, then equation (20) is replaced by

$$\text{Var}(Q)/\overline{Q}^2 = \frac{2W_0}{\Delta} \left[\frac{1}{\kappa}(e^\kappa - 1) - 1 \right] \quad (24)$$

where now

$$\kappa \equiv (2 - 2^{\beta-1}) \ln \frac{1}{\overline{Q}} \equiv (2 - 2^{\beta-1}) \overline{\tau} \quad (25)$$

(cf. equations 17 and 18). When the mean optical depth $\bar{\tau}$ is not too large, we have approximately

$$\text{Var}(Q)/\bar{Q}^2 \approx (2 - 2^{\beta-1}) \frac{W_0}{\Delta} \bar{\tau} \quad (26)$$

A typical value for β is 1.5 (Sargent 1988a; we will have more to say about this value in Paper II).

Let us now compare these results to the SSG data. We estimate $\text{Var}(Q)/\bar{Q}^2$ in the data by the following procedure, which is designed to greatly reduce the variance due to errors in the continuum fits: For each measurement Q_i , we estimate $Q_i - \bar{Q}$ not by subtracting the overall mean \bar{Q} , but rather by subtracting a local weighted average of the Q_i 's for that particular quasar and neighboring wavelength bins. The weighted average is taken to have a triangular profile, with unit amplitude at bin i falling to zero at bins $i \pm 30$ (that is, $\pm 300\text{\AA}$ in observed wavelength). With a mean redshift of about $1 + z = 4.5$, this means that we are sensitive to variations in the extrapolated continuum only over scales of 65 \AA , which are likely negligible. Our measured result for the SSG sample is

$$\text{Var}(Q)/\bar{Q}^2 = 0.05 \pm 0.01 \quad (27)$$

where the $1\text{-}\sigma$ error bar is again obtained by resampling.

At a redshift $1 + z = 4.5$ we have $\bar{\tau} = 0.67$ (equation 5; cf. Figure 4). Using SSG's quoted spectral resolution $\Delta = 25\text{\AA}$, equation (24) implies $W_0 \approx 0.7\text{\AA}$, or $b \sim 80\text{km s}^{-1}$. This value is implausibly large by about a factor of about 2. At this stage of analysis there are three possible resolutions: (1) Since we do not have an absolute calibration of the measurement noise for each data point, we cannot say with confidence that this excess variance is not an instrumental effect. (2) The excess variance might be due to a two-point correlation function ξ in the clouds on velocity scales comparable to the instrumental resolution Δ , that is $\Delta c/1216\text{\AA} \sim 6000\text{km s}^{-1}$. Clustering of Lyman α clouds has previously been detected only at much smaller scales (e.g., Webb 1987, Crotts 1989). (3) The factor of 2 discrepancy might be an artifact of our simplistic analytic assumption of a square line profile; the observed fluctuations might in fact be consistent with a Poisson random distribution of clouds.

In Paper II, further analysis will show that the third resolution is the most likely one. We note here that our finding fluctuations comparable to the predicted Poisson value helps confirm an implicit assumption that we have made throughout this paper: the overall absorption at high redshifts is *not* significantly due to a continuous Gunn-Peterson trough rather than a superposition of individual clouds.

6. Conclusions

The principal results of this paper are a set of techniques for the statistical analysis of the Lyman α forest that can be used without the necessity of resolving and identifying individual lines. We have seen that there exist a set of well-defined parameters that can thus be measured with useful accuracy. Where there is overlap with previous results (including those obtained by high resolution studies) our methods give a reassuring agreement. Additionally, our methods make accessible some new parameters, such as the equivalent width ratios of the Lyman series.

For the redshift range of about $2.5 < z_{abs} < 4.3$ that is sampled by the lines of sight to 29 SSG quasars, we have the following specific results:

1. The mean Lyman α absorption at a redshift z is well approximated by $\bar{\tau}(z) = A(1+z)^{\gamma+1}$ with $\gamma = 2.46 \pm 0.37$ and $A = 0.0175 - 0.0056\gamma \pm 0.0002$, in agreement with previous determinations at moderate redshifts from high spectral resolution, ground-based observations (though not with the low redshift measurements of HST).

2. The ratio of mean Lyman β absorption to mean Lyman α absorption at a fixed absorption redshift (which is diagnostic of the physical state of the clouds at that redshift) is 0.476 ± 0.054 . The ratios for Lyman γ , δ , and ϵ are also measurable, and given, along with the error covariance matrix for all the ratios, in Table 3.

3. Fluctuations around the mean absorption are comparable to, possibly a factor of 2 larger than can be explained as, simple Poisson fluctuations in the number of clouds along the line of sight. The factor of 2 discrepancy could be due to measurement noise, model imprecision (see Paper II), or a small, positive two-point correlation function ξ on a scale of 6000 km s^{-1} . The measurement of fluctuations comparable to the Poisson value argues against a significant Gunn-Peterson trough at high redshifts.

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A. Appendix A

We here derive some statistical properties of the extinction optical depth function $\tau(\lambda)$ as a function of wavelength. We base our results on a simple statistical model, closely

related to random line models. Anticipating needs of future papers, our development is somewhat more general than strictly required for the present applications.

We first express $\tau(\lambda)$ as a sum of contributions from a number of clouds along the line of sight,

$$\tau(\lambda) = \sum_i T(\lambda|\lambda_i, \mathbf{p}_i), \quad (\text{A1})$$

where $T(\lambda|\lambda_i, \mathbf{p}_i)$ is the contribution from the i th cloud. This depends on the central wavelength of the cloud, λ_i , as well as on a vector of parameters \mathbf{p}_i for the cloud, which may include its column density N_i , its velocity parameter b_i , and possibly other parameters.

It may be helpful at this point to remark that the particular form for T used in the body of the paper is $T(\lambda|\lambda_i, \mathbf{p}_i) = N_i \alpha(\lambda - \lambda_i, b_i)$, where N_i is the column density, b_i the velocity parameter, and $\alpha(\lambda - \lambda_i, b_i)$ is the atomic absorption coefficient, typically a Voigt function. In this special case λ_i simply defines the central wavelength of the profile, and the parameters \mathbf{p}_i consist only of N_i and b_i . We shall generally assume that the λ dependence of T falls off sufficiently rapidly away from λ_i to assure the convergence of certain integrals.

It is easiest to express our statistical model as a two-step process: First of all, we assume that the positions of the central wavelengths λ_i of the clouds are Poisson random distributed with a given mean density of lines per unit wavelength interval. Second, given the positions $\lambda_1, \lambda_2, \dots$, of the clouds, the parameters $\mathbf{p}_1, \mathbf{p}_2, \dots$ are assumed to be independently random, that is, their joint conditional distribution function is a product of independent factors, one for each cloud,

$$P(\mathbf{p}_1, \mathbf{p}_2, \dots | \lambda_1, \lambda_2, \dots) = \prod_i P(\mathbf{p}_i | \lambda_i). \quad (\text{A2})$$

Thus the distribution of the parameters \mathbf{p}_i of the i th cloud can depend on the associated λ_i , but not on the \mathbf{p} 's or λ 's of any other clouds.

Let the mean density of line centers per unit wavelength interval and in a volume $d\mathbf{p}$ of parameter space be given by $\mathcal{N}(\lambda, \mathbf{p})d\mathbf{p}$. Generalizing the discussion in §5, we have here allowed the line density function \mathcal{N} to depend on wavelength. For convenience in the following derivations we shall assume that the mean total density of lines at λ ,

$$n(\lambda) = \int \mathcal{N}(\lambda, \mathbf{p}) d\mathbf{p}, \quad (\text{A3})$$

is finite, temporarily ignoring the possibility (alluded to in the text) of a power law divergence at small column densities. We may then take

$$P(\mathbf{p}_i | \lambda_i) = \frac{\mathcal{N}(\lambda_i, \mathbf{p}_i)}{n(\lambda_i)}, \quad (\text{A4})$$

as the conditional probability density for the parameters \mathbf{p}_i for the i th cloud.

The statistical model defined above is similar to the well-known *Poisson* or *shot process* (see, e.g., Parzen 1962). However, this process has been generalized here to allow the mean density of centers to vary with λ and the function T to depend on λ and λ_i in more complicated ways than simply through their difference. This generalization is not strongly needed for the applications of this paper, but will play a more important role for later papers in this series.

A very powerful and general way to express the statistical properties of the function $\tau(\lambda)$ is through its *characteristic functional* (see, e.g., Bartlett 1955). This is defined for an arbitrary function $\mu(\lambda)$ by

$$\Phi[\mu] \equiv \left\langle \exp \left[i \int \mu(\lambda) \tau(\lambda) d\lambda \right] \right\rangle, \quad (\text{A5})$$

where the angle brackets denote averaging over the distribution of line centers and over the parameter space \mathbf{p} for each line.

The range of integration in equation (A5) has been left unspecified. In principle we would like it to be the infinite range of λ , but for purposes of the following derivation it is convenient to choose it temporarily to be some definite, large (but finite) interval, which adequately spans the essential nonzero region of the function $\mu(\lambda)$. Such a finite region of integration implies that the mean number of clouds in the interval

$$\overline{M} = \int n(\lambda) d\lambda = \int d\lambda \int d\mathbf{p} \mathcal{N}(\lambda, \mathbf{p}). \quad (\text{A6})$$

is finite. The *actual* number of lines M in the interval is a random variable with Poisson distribution function

$$P_M = \frac{\overline{M}^M}{M!} e^{-\overline{M}}, \quad (\text{A7})$$

which can range from 0 to ∞ . For each realization of M , the summation in equation (A1) is from $i = 1$ to M .

We may now evaluate the part of the averaging in equation (A5) due to the Poisson distribution of line centers. We note that the probability that a line occurs in a differential interval $d\lambda_i$ about λ_i is simply $d\lambda_i n(\lambda_i)/\overline{M}$, and that the probability distribution for each line is independent of all other lines (this is the Poisson assumption). Then

$$\Phi[\mu] = \sum_{M=0}^{\infty} \frac{\overline{M}^M}{M!} e^{-\overline{M}} \left\langle \int d\lambda_1 \frac{n(\lambda_1)}{\overline{M}} \cdots \int d\lambda_M \frac{n(\lambda_M)}{\overline{M}} \exp \left[i \sum_{i=1}^M \int \mu(\lambda) T(\lambda | \lambda_i, \mathbf{p}_i) d\lambda \right] \right\rangle, \quad (\text{A8})$$

where now the angular brackets refer only to the average over all the parameters $\mathbf{p}_1, \mathbf{p}_2, \dots$. With equations (A2) and (A4), we have,

$$\begin{aligned}\Phi[\mu] &= e^{-\overline{M}} \sum_{M=0}^{\infty} \frac{\overline{M}^M}{M!} \prod_{i=1}^M \left\langle \int d\lambda_i \frac{n(\lambda_i)}{\overline{M}} \exp \left[i \int \mu(\lambda) T(\lambda|\lambda_i, \mathbf{p}_i) d\lambda \right] \right\rangle_i \\ &= e^{-\overline{M}} \sum_{M=0}^{\infty} \frac{1}{M!} \prod_{i=1}^M \int d\lambda_i n(\lambda_i) \int d\mathbf{p}_i \frac{\mathcal{N}(\lambda_i, \mathbf{p}_i)}{n(\lambda_i)} \exp \left[i \int \mu(\lambda) T(\lambda|\lambda_i, \mathbf{p}_i) d\lambda \right] \quad (\text{A9})\end{aligned}$$

We note that all factors in the product are identical, since the dummy variable of integration is irrelevant, so the product reduces to a simple power of one factor, which we may choose to be the one for $i = 1$,

$$\Phi[\mu] = e^{-\overline{M}} \sum_{M=0}^{\infty} \frac{1}{M!} \left\{ \int d\lambda_1 \int d\mathbf{p}_1 \mathcal{N}(\lambda_1, \mathbf{p}_1) \exp \left[i \int \mu(\lambda) T(\lambda|\lambda_1, \mathbf{p}_1) d\lambda \right] \right\}^M. \quad (\text{A10})$$

Then using equation (A6) and the power series for the exponential, along with some changes in dummy variables of integration, we find

$$\begin{aligned}\Phi[\mu] &= \left\langle \exp \left[i \int \mu(\lambda') \tau(\lambda') d\lambda' \right] \right\rangle \\ &= \exp \left\{ - \int d\lambda \int d\mathbf{p} \mathcal{N}(\lambda, \mathbf{p}) \left(1 - \exp \left[i \int \mu(\lambda') T(\lambda'|\lambda, \mathbf{p}) d\lambda' \right] \right) \right\}. \quad (\text{A11})\end{aligned}$$

Recall that finite limits of integration were introduced to make the quantity \overline{M} finite. However, assuming a sufficiently localized $\mu(\lambda)$, such divergent quantities no longer appear in equation (A11), so the integrals in this formula can be considered to have arbitrary limits. Similarly, we note that the quantity $n(\lambda)$ no longer appears, so in many cases this equation can have meaning even when the integral in equation (A3) diverges.

Equation (A11) is the major result describing the statistical properties of the underlying Poisson process. We can use it to derive a variety of important statistical results. For example, if the two sides of the equation are expanded to first order in $\mu(\lambda')$ and the corresponding terms equated, one obtains the result for the mean of $\tau(\lambda')$,

$$\langle \tau(\lambda') \rangle = \int d\lambda \int d\mathbf{p} \mathcal{N}(\lambda, \mathbf{p}) T(\lambda'|\lambda, \mathbf{p}). \quad (\text{A12})$$

Equation (A11) can also be expanded to second order. In doing so, one must be careful to introduce a new dummy variable of integration λ'' on one factor. Omitting details, we obtain the correlation function,

$$\langle [\tau(\lambda') - \langle \tau(\lambda') \rangle] [\tau(\lambda'') - \langle \tau(\lambda'') \rangle] \rangle = \int d\lambda \int d\mathbf{p} \mathcal{N}(\lambda, \mathbf{p}) T(\lambda'|\lambda, \mathbf{p}) T(\lambda''|\lambda, \mathbf{p}). \quad (\text{A13})$$

This result generalizes the well-known Campbell’s theorem (see, e.g., Parzen 1962) by allowing λ dependence in the statistical quantities.

For the study of Lyman α clouds, more important than averages involving τ itself are averages involving the exponential extinction law,

$$q(\lambda) = e^{-\tau(\lambda)}. \quad (\text{A14})$$

Setting $\mu(\lambda') = i\delta(\lambda' - \lambda)$ in equation (A11) gives immediately,

$$\bar{q}(\lambda) = \langle q(\lambda) \rangle = \exp \left(- \int d\lambda' \int d\mathbf{p} \mathcal{N}(\lambda, \mathbf{p}) \left[1 - e^{-T(\lambda'|\lambda, \mathbf{p})} \right] \right), \quad (\text{A15})$$

This formula represents a generalization of equation (18) of the text.

Correlation properties of $q(\lambda)$ can be found similarly. Setting $\mu(\lambda''') = i\delta(\lambda''' - \lambda') + i\delta(\lambda''' - \lambda'')$ in equation (A11) gives

$$\langle q(\lambda') q(\lambda'') \rangle = \exp \left[- \int d\lambda''' \int d\mathbf{p} \mathcal{N}(\lambda''', \mathbf{p}) \left(1 - e^{-T(\lambda'|\lambda''', \mathbf{p}) - T(\lambda''|\lambda''', \mathbf{p})} \right) \right]. \quad (\text{A16})$$

[We note that a special case of equation (A16), for $\lambda' = \lambda''$, and for a homogeneous model, was given by Møller, P., and Jakobsen (1990)]. Using equation (A15) twice, this can be written

$$\langle q(\lambda') q(\lambda'') \rangle = \bar{q}(\lambda') \bar{q}(\lambda'') e^{-H(\lambda', \lambda'')}, \quad (\text{A17})$$

where

$$H(\lambda', \lambda'') = \int d\lambda''' \int d\mathbf{p} \mathcal{N}(\lambda''', \mathbf{p}) \left(1 - e^{-T(\lambda'|\lambda''', \mathbf{p})} \right) \left(1 - e^{-T(\lambda''|\lambda''', \mathbf{p})} \right). \quad (\text{A18})$$

We may also write

$$\langle [q(\lambda') - \bar{q}(\lambda')] [q(\lambda'') - \bar{q}(\lambda'')] \rangle = \bar{q}(\lambda') \bar{q}(\lambda'') \left(e^{-H(\lambda', \lambda'')} - 1 \right). \quad (\text{A19})$$

Note that $H(\lambda', \lambda'')$ is small when the wavelength difference $|\lambda' - \lambda''|$ is much larger than the atomic line widths, since then the T functions in equation (A18) do not significantly overlap. Thus equation (A19) shows that the values of $q(\lambda)$ are essentially uncorrelated for such wavelength differences.

B. Appendix B

The purpose of this Appendix is to derive equations (20) through (25) of §5, applying the general results of Appendix A.

Before proceeding, we want to take into account that the measured extinction is not $q(\lambda)$, but rather an average over the instrumental profile,

$$Q(\lambda) = \int w(\lambda - \lambda') q(\lambda') d\lambda'. \quad (\text{B20})$$

It is usually sufficiently accurate to take w to be a function of wavelength differences $\lambda - \lambda'$. In this Appendix, though not in the main text, it is defined with normalization

$$1 = \int w(\lambda - \lambda') d\lambda'. \quad (\text{B21})$$

Besides $Q(\lambda)$ itself, we are also interested in its square

$$Q^2(\lambda) = \int d\lambda' w(\lambda - \lambda') \int d\lambda'' w(\lambda - \lambda'') q(\lambda') q(\lambda''), \quad (\text{B22})$$

which is needed in defining the variance of Q . A useful generalization of this is the product of Q 's at two different points,

$$Q(\lambda_1) Q(\lambda_2) = \int d\lambda' w(\lambda_1 - \lambda') \int d\lambda'' w(\lambda_2 - \lambda'') q(\lambda') q(\lambda''), \quad (\text{B23})$$

which is needed in defining the correlation properties of Q . (The variables λ_1 and λ_2 here are general variables, and have nothing to do with the $i = 1$ and $i = 2$ clouds.)

We shall now assume that the distribution \mathcal{N} is independent of λ (or at least that its scale of variation in λ is much larger than either the atomic line profile or the instrumental resolution). We also assume that $T(\lambda|\lambda_i, \mathbf{p}_i) = N_i \alpha(\lambda - \lambda_i, b_i)$, which depends on λ and λ_i only through their difference $\lambda - \lambda_i$, at least locally.

Substituting this expression for T into equation (A15), we immediately obtain,

$$\langle q(\lambda_0) \rangle = \langle e^{-\tau(\lambda_0)} \rangle = \exp \left(- \int W(\mathbf{p}) \mathcal{N}(\mathbf{p}) d\mathbf{p} \right), \quad (\text{B24})$$

where

$$W(\mathbf{p}) = \int d\lambda \{1 - \exp[-N\alpha(\lambda, b)]\}. \quad (\text{B25})$$

is the equivalent width of a line with column density N and velocity parameter b . In deriving equation (A13), we have shifted the variable of integration, showing that $W(\mathbf{p})$ is independent of λ_0 (at least over wavelength intervals sufficiently small that α can be considered to be only a function of wavelength differences). Consequently $\langle q \rangle$ is also

independent of λ_0 . Then from equations (B20) and (B21), it follows that $\overline{Q} \equiv \langle Q \rangle = \langle q \rangle$. Thus equations (B24) and (B25) constitute an independent proof of equation (18) of the text.

We next consider the problem of determining the statistical average of Q^2 , given in equation (B22). Actually, it is almost as easy to determine the average of the product of Q 's in equation (B23), so we shall do this. First of all, we need the average of $\exp[-\tau(\lambda') - \tau(\lambda'')]$. This is given by equation (A19), which can now be written

$$\langle [q(\lambda') - \overline{Q}] [q(\lambda'') - \overline{Q}] \rangle = \overline{Q}^2 (e^{-H(\lambda'' - \lambda')} - 1). \quad (\text{B26})$$

where

$$H(\lambda) = \int d\mathbf{p} \mathcal{N}(\mathbf{p}) \int d\lambda''' (1 - e^{-N\alpha(\lambda''', b)}) (1 - e^{-N\alpha(\lambda''' + \lambda, b)}). \quad (\text{B27})$$

Here we have made appropriate changes of variables to take advantage of the difference dependence of the atomic absorption coefficients. Using equation (B23), we find that

$$\langle [Q(\lambda_1) - \overline{Q}] [Q(\lambda_2) - \overline{Q}] \rangle = \overline{Q}^2 \int d\lambda' w(\lambda_1 - \lambda') \int d\lambda'' w(\lambda_2 - \lambda' + \lambda'') (e^{-H(\lambda'')} - 1). \quad (\text{B28})$$

Equation (B28) can be substantially simplified under conditions where the individual atomic lines are very much under-resolved, that is, when the instrumental function W is much broader than atomic widths. (This is true for the data of the present work.) In this case, the effective range of the λ'' integration is very much smaller than the width of the instrumental profile, and we can ignore the λ'' dependence in the function $w(\lambda_1 - \lambda' + \lambda'')$, setting it equal to $w(\lambda_1 - \lambda')$ and taking it outside the λ'' integration. This gives

$$\langle [Q(\lambda_1) - \overline{Q}] [Q(\lambda_2) - \overline{Q}] \rangle = \overline{Q}^2 \left[\int d\lambda'' w(\lambda'') w(\lambda_2 - \lambda_1 + \lambda'') \right] \int d\lambda'' (e^{-H(\lambda'')} - 1). \quad (\text{B29})$$

The quantity in brackets determines the dependence of the correlation function on the wavelength difference $\lambda_1 - \lambda$. When $\lambda_1 = \lambda$, the LHS of this equation is equal to the variance of Q , denoted $\text{Var}(Q)$. Thus

$$\text{Var}(Q) = \overline{Q}^2 \left[\int d\lambda'' w^2(\lambda'') \right] \int d\lambda'' (e^{-H(\lambda'')} - 1). \quad (\text{B30})$$

Thus equation (B29) can be written in the simple form

$$\langle [Q(\lambda_1) - \overline{Q}] [Q(\lambda_2) - \overline{Q}] \rangle = \text{Var}(Q) Y(\lambda_2 - \lambda_1), \quad (\text{B31})$$

where

$$Y(\lambda) = \frac{\int d\lambda'' w(\lambda'') w(\lambda + \lambda'')}{\int d\lambda'' w^2(\lambda'')} \quad (\text{B32})$$

For the data of this paper, the instrumental function is well represented as the rectangular function,

$$w(\lambda) = \begin{cases} \Delta^{-1}, & \lambda < \Delta/2, \\ 0, & \lambda > \Delta/2. \end{cases} \quad (\text{B33})$$

In this case,

$$Y(\lambda) = (1 - |\lambda|/\Delta)_+ \quad (\text{B34})$$

where $(\dots)_+$ implies zero for negative arguments. This shows that the correlation function has a central value equal to the variance, and has a triangular shape, going to zero for lags greater than the instrumental width.

It remains only to determine the variance $\text{Var}(Q)$ itself. For the rectangular instrumental profile this is given by

$$\text{Var}(Q)/\overline{Q}^2 = \frac{1}{\Delta} \int d\lambda'' \left(e^{-H(\lambda'')} - 1 \right). \quad (\text{B35})$$

In general the integral in equation (B35) must be evaluated numerically, which also involves numerical evaluation of the function $H(\lambda)$ defined in equation (B27). However, considerable insight can be obtained by investigating the simple model using the single rectangular line profile,

$$\alpha(\lambda - \lambda', b) = \begin{cases} \alpha_0, & |\lambda - \lambda'| < W_0/2, \\ 0, & |\lambda - \lambda'| > W_0/2, \end{cases} \quad (\text{B36})$$

where α_0 is the height of the profile and W_0 is its full-width. The parameter b does not appear here, since it implicitly takes a single value, which sets the values of the constants α_0 and W_0 . The parameter vector \mathbf{p} then consists only of the column density N .

With this choice of profile equation (B25) gives

$$W(N) = W_0 \left(1 - e^{-N\alpha_0} \right), \quad (\text{B37})$$

and equation (B24) then gives

$$\overline{Q} = \exp \left[-W_0 \int dN \mathcal{N}(N) \left(1 - e^{-N\alpha_0} \right) \right]. \quad (\text{B38})$$

Having found the mean transmission, we next want to find the variance (and correlation function). From equation (B27) it follows that,

$$H(\lambda) = \kappa (1 - |\lambda|/W_0)_+, \quad (\text{B39})$$

where

$$\kappa = W_0 \int dN \mathcal{N}(N) \left(1 - e^{-N\alpha_0}\right)^2 \quad (\text{B40})$$

The variance is then found from equation (B35),

$$\text{Var}(Q)/\overline{Q}^2 = \frac{2}{\Delta} \int_0^{W_0} d\lambda \left(e^{-\kappa(1-\lambda/W_0)} - 1\right) = \frac{2W_0}{\Delta} \left[\frac{1}{\kappa} (e^\kappa - 1) - 1\right], \quad (\text{B41})$$

The simplest special case of these results is when there is also a single column density N_0 for all clouds. In this case $\mathcal{N}(N) = n\delta(N - N_0)$, so that equations (B37) and (B38) imply that the mean transmission is

$$\overline{Q} = \exp[-n\overline{W}], \quad (\text{B42})$$

where the equivalent width of each line is,

$$\overline{W} = W_0 \left(1 - e^{-N_0\alpha_0}\right). \quad (\text{B43})$$

The variance is given by (B41), where now,

$$\kappa = nW_0 \left(1 - e^{-N\alpha_0}\right)^2 = \frac{n\overline{W}^2}{W_0}, \quad (\text{B44})$$

which proves equation (21) of the text.

We can also find results for the case where \mathcal{N} is the power law in N , $\mathcal{N}(N) = KN^{-\beta}$. Then

$$\overline{Q} = \exp[-n\overline{W}], \quad (\text{B45})$$

where

$$\begin{aligned} \overline{W} &= KW_0 \int_0^\infty dN N^{-\beta} \left(1 - e^{-N\alpha_0}\right) \\ &= \frac{KW_0\alpha_0}{\beta - 1} \int_0^\infty dN N^{1-\beta} e^{-N\alpha_0} \\ &= KW_0\alpha_0^{\beta-1} \Gamma(2 - \beta)/(\beta - 1) \end{aligned} \quad (\text{B46})$$

An integration by parts has been used, along with the definition of the Γ function.

Similarly, we now find from equation (B40),

$$\begin{aligned} \kappa &= nKW_0 \int_0^\infty dN N^{-\beta} \left(1 - e^{-N\alpha_0}\right)^2 \\ &= \frac{2nKW_0\alpha_0}{\beta - 1} \int_0^\infty dN N^{1-\beta} \left(e^{-N\alpha_0} - e^{-2N\alpha_0}\right) \\ &= (2 - 2^{\beta-1})nKW_0\alpha_0^{\beta-1} \Gamma(2 - \beta)/(\beta - 1) \end{aligned} \quad (\text{B47})$$

Using equation (B46) we now find

$$\kappa = (2 - 2^{1-\beta})n\overline{W} = (2 - 2^{\beta-1})\ln \frac{1}{\overline{Q}}, \quad (\text{B48})$$

which proves equation (25) of the text.

REFERENCES

- Bahcall, J.N., Jannuzi, B.T., Schneider, D.P., Hartig, G.F. and Green, R.F. 1992, ApJ, 397, 68.
- Bahcall, J.N., Bergeron, J., Boksenberg, A., Hartig, G.F., Jannuzi, B.T., Kirhakos, S., Sargent, W.L.W., Savage, B.D., Schneider, D.P., Turnshek, D.A., Weymann, R.J., and Wolfe, A.M. 1993, ApJS, in press.
- Bajtlik, S., Duncan, R.C., and Ostriker, J.P. 1988, ApJ, 327, 570.
- Bartlett, M. S. 1955, *An Introduction to Stochastic Processes*, (Cambridge: Camb. Univ. Press)
- Bechtold, J., Green, R.F., Weymann, R.J., Schmidt, M., Estabrook, F.B., Sherman, R.D., Wahlquist, H.D., and Heckman, T.M. 1984, ApJ, 281, 76.
- Carswell, R.F., Lanzetta, K.M., Parnell, H.C., and Webb, J.K. 1991, ApJ, 371, 36.
- Crotts, A.P.S., 1989, ApJ, 336, 550.
- Efron, B. 1982, *The Jackknife, the Bootstrap, and Other Resampling Plans* (Philadelphia: S.I.A.M.).
- Efron, B., and Tibshirani, R. 1986, Statistical Science, 1, 54.
- Francis, P.J., Hewett, P.C., Foltz, C.B., Chaffee, F.H., Weymann, R.J., and Morris, S.L. 1991, ApJ, 373, 465.
- Giallongo, E., and Cristiani, S. 1990, MNRAS, 247, 696.
- Goody, R.M. 1964, *Atmospheric Radiation. I. Theoretical Basis* (Oxford: Oxford University Press).
- Jenkins, E.B., and Ostriker, J.P. 1991, ApJ, 376, 33.
- Lynds, C.R. 1971, ApJ, 164, L73.
- Meyer, D.M., and York, D.G. 1987, ApJ, 315, L5.
- Møller, P., and Jakobsen, P. 1990, A&A, 228, 299.
- Murdoch, H.S., Hunstead, R.W., Pettini, M., and Blades, J.C. 1986, ApJ, 309, 19.
- Oke, J.B., and Korycansky, D.G. 1982, ApJ, 255, 11.
- Parzen, E. 1962, *Stochastic Processes*, (San Francisco: Holden-Day)
- Pettini, M., Hunstead, R.W., Smith, L.J., and Mar, D.P. 1990, MNRAS, 246, 545.
- Press, W.H., Teukolsky, S.A., Vetterling, W.T., and Flannery, B.P. 1992, *Numerical Recipes in FORTRAN*, 2nd ed. (New York: Cambridge University Press), 686.

- Richstone, D.O., and Schmidt, M. 1980, ApJ, 235, 361.
- Rybicki, G.B., and Press, W.H. 1992, ApJ, 398, 169.
- Sargent, W.L.W., Young, P.J., Boksenberg, A., and Tytler, D. 1980, ApJS, 42, 41.
- Sargent, W.L.W. 1988a, in *QSO Absorption Lines: Probing the Universe*, J.C. Blades, D. Turnshek, C.A. Norman, eds. (Cambridge: Cambridge University Press).
- Sargent, W.L.W. 1988b, in *Large Scale Structures of the Universe*, IAU Symposium 130, J. Audouze, M.-C. Pelletan, A. Szalay, eds. (Dordrecht: Kluwer).
- Schneider, D.P., Schmidt, M., and Gunn, J.E. 1989, AJ, 98, 1507.
- Schneider, D.P., Schmidt, M., and Gunn, J.E. 1989, AJ, 98, 1951.
- Schneider, D.P., Schmidt, M., and Gunn, J.E. 1991, AJ, 101, 2004.
- Steidel, C.C., and Sargent, W.L.W. 1987, ApJ, 313, 171.
- Warren, S.J., Hewitt, P.C., and Osmer, P.S. 1991, ApJS, 76, 23..
- Webb, J.K. 1987, in *Observational Cosmology*, Proc. IAU Symposium 124, A.Hewitt, ed. (Dordrecht: Reidel).
- Webb, J.K., Barcons, X., Carswell, R.F., and Parnell, H.C. 1992, MNRAS, 255, 319.

<i>Line</i>	λ_0 (Å)	w_0 (Å)
Ly α	1216.	20.
NV	1240.	18.
OI	1302.	7.2
CII	1335.	6.0
SiIV/OIV	1400.	11.
CIV	1549.	15.
HeII/OIII	1651.	77.
FeII feature	1770.	26.
AlIII	1858.	30.
CIII	1909.	22.

Table 2: Lines, emission wavelengths, and nominal widths used for fitting the underlying continua of individual quasars in the range 1250 Å to 2200 Å. Nominal widths are used only as the starting point for a linearized width correction.

<i>Ratio</i> $i =$	$\widehat{W}_\beta/\widehat{W}_\alpha$ 1	$\widehat{W}_\gamma/\widehat{W}_\alpha$ 2	$\widehat{W}_\delta/\widehat{W}_\alpha$ 3	$\widehat{W}_\epsilon/\widehat{W}_\alpha$ 4
<i>Measured Value</i>	0.476	0.351	0.157	0.118
<i>Standard Error (1-σ)</i>	0.054	0.072	0.082	0.072
<i>Correlation Matrix C_{ij}</i>				
$j = 1$	0.00296	−0.00156	−0.00005	0.00085
$j = 2$	−0.00156	0.00532	−0.00266	−0.00041
$j = 3$	−0.00005	−0.00266	0.00668	−0.00351
$j = 4$	0.00085	−0.00041	−0.00351	0.00522
<i>Inverse Correlation Matrix C_{ij}^{-1}</i>				
$j = 1$	422.	147.	49.5	−23.6
$j = 2$	147.	355.	224.	155.
$j = 3$	49.5	224.	378.	264.
$j = 4$	−23.6	155.	264.	385.
<i>Coefficient of Correlation r_{ij}</i>				
$j = 1$	1.00	−0.39	−0.01	0.22
$j = 2$	−0.39	1.00	−0.45	−0.08
$j = 3$	−0.01	−0.45	1.00	−0.59
$j = 4$	0.22	−0.08	−0.59	1.00

Table 3: Measured quantities associated with the ratios of mean equivalent widths for Lyman α , β , γ , δ , and ϵ in the SSG sample. In the column headings $\widehat{W}_\alpha \equiv W_\alpha/\lambda_\alpha$, $\widehat{W}_\beta \equiv W_\beta/\lambda_\beta$, etc.

Fig. 1.— The observed spectra of 29 SSG quasars are here superposed after shifting each to its emission rest frame and scaling each to a common magnitude at 1450 Å. Despite indisputable differences in the individual quasars’ continuum slopes and emission features, there is considerable similarity in the spectra. The principal interest of this paper is in the statistical analysis of the Lyman α forest shortward of 1200 Å.

Fig. 2.— Fitted line profiles and derived continuum models for each of the 29 SSG quasars analyzed. Fitting is done by a linear model, which can give artifacts in line wings, but is otherwise more robust than a general nonlinear fit (see text). The purpose of these fits is to obtain continuum models that can be extrapolated shortward of 1200 Å.

Fig. 3.— Extrapolations of the continuum models shown in Figure 2 to emitted wavelengths between 930 Å and 1200 Å. Each point shown, taken as a fraction of the extrapolated continuum above it, is an (approximately independent) measurement of Lyman α absorption at a calculable redshift. The ensemble of all the points in this Figure (excluding a small range of emitted wavelengths around Lyman β) is the data set that is analysed in the rest of this paper.

Fig. 4.— Left, the subset of points from Figure 3 with emitted wavelength between 1050 Å and 1170 Å (each normalized to its extrapolated continuum level) is plotted as a function of emitted wavelength. Since the 29 SSG quasars vary substantially in redshift, the observed transmission varies widely, with no significant trend. Right, the same data is plotted as a function of observed wavelength or, equivalently, absorption redshift for Lyman α (top scale). Here the trend with redshift is clear. The solid line fits a power law model with mean optical depth varying as $(1+z)^{\gamma+1}$, with $\gamma = 1.46$. The shaded band approximates the range of statistical uncertainty in the fit, as determined by the bootstrap method of resampling the 29 quasars (see text).

Fig. 5.— The individual transmission measurements are shown as a function of emitted wavelength, after correcting each point for Lyman α absorption at its own absorption redshift. The shaded band is a moving average of the points. One sees the Ly- β , Ly- γ , Ly- δ , and possibly Ly- ϵ decrements. These are jointly fitted, after shifting each point in the proper emission region to its proper absorption redshift, and equivalent width ratios for the corresponding Lyman lines, along with an error covariance matrix (again obtained by the bootstrap method), are obtained. See text for details. These mean equivalent width ratios place significant constraints on physical conditions in the clouds.